Journal of Engineering Mathematics, Vol. 9, No. 3, July 1975
Noordhoff International Publishing–Leyden
Printed in The Netherlands

One-dimensional wave propagation and Fokker–Planck's equation

E. GHANDOUR

Department of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, Israel

(Received August 5, 1974)

SUMMARY

The one-dimensional random wave propagation problem is analyzed. The medium is assumed to be characterized by a stationary index of refraction of the white-Gaussian process. By considering the initial value equation for the boundary value stochastic Green's function, the Fokker–Planck equation for the density function of the amplitude and phase of the reflected wave is constructed. By employing an averaging theorem due to Khas'minskii, the mean power reflected and transmitted is obtained.

1. Introduction

In this paper we shall examine the propagation of scalar waves through a one-dimensional random medium, the medium being characterized by a stationary index of refraction of the white-Gaussian type. We shall assume the wave field to be harmonically time dependent. We shall obtain expressions which would determine the scattering properties of the refracting medium, and in particular, the mean power reflected and transmitted. The formulation shall be carried out by constructing the Fokker–Planck equation from the appropriate initial value equations for the density function of the amplitude and phase of the wave field, and then apply a theorem of Khas'minskii for the limit behavior of the solution of a parabolic equation whose coefficients oscillate rapidly [1].

The beginning of our analysis will be based on results obtained in [2] and [3] in which the boundary value problem for the scattering region has been transformed into a Cauchy type initial value problem for the random boundary values of the Green's function. The idea of transforming the boundary value problem into an initial value problem and proceeding to the Fokker–Planck equation was motivated by [4], [5]. One of the main aims in this paper is to elucidate the applicability of this approach to stochastic boundary value problems and apply the averaging theorem.

The propagation of waves through random media has been examined by various approaches in [3, 6–10], the latter differing from the first five in that the media is considered discrete instead of continuous. The work in [8] is closely related to our work in that use is made of another theorem of Khas'minskii for the limiting solutions of stochastic differential equations with a small parameter [11], [12]. Ours is a more restricted problem in that in the latter work the solutions obtained are for a broader class of random media.

2. Statement of the problem

Let a plane scalar wave propagate in the region $x > \lambda$ onto a bounded random medium contained in $[0, \lambda]$. For time harmonic dependence $e^{-i\omega t}$ of the wave field $w(x)$, it follows that $w(x)$ satisfies the equation

$$L[w] = w''(x) + k^2(x)w(x) = 0\,, \qquad -\infty < x < \infty\,, \tag{1}$$

where the index of refraction $k^2(x)$ is given by

$$k^2(x) = \begin{cases} k_0^2\,, & x < 0\,, \\ k_0^2(1+\beta\gamma(x;\mu))\,, & 0 < x < \lambda\,, \\ k_0^2\,, & x > \lambda\,, \end{cases} \tag{2}$$

* This research was supported in part by the European Research Office under contract DAJA37-73-R-0242.

and β is a parameter defining the strength of the fluctuations in the scattering region. $\gamma(x;\mu)$, $\mu\in\Omega$, is a real valued stochastic process defined on the probability space (Ω, S, P). Both w and $w_{,x}$ are continuous across the boundaries $x=0$ and $x=\lambda$. These continuity conditions together with Eqns. (1) and (2) constitute the appropriate boundary value problem.

Let $e^{-ik_0(x-\lambda)}$ represent the incident wave in $x>\lambda$. By utilizing the radiation condition for $|x|\to\infty$, we can write the solutions outside of the scattering region as

$$w(x;\mu)=e^{-ik_0(x-\lambda)}+R(\lambda;\mu)e^{ik_0(x-\lambda)}, \quad x>\lambda, \tag{3a}$$

$$w(x;\mu)=T(\lambda;\mu)e^{-ik_0x}, \quad x<0. \tag{3b}$$

where R and T are the reflected and transmitted amplitudes, respectively. They are complex valued random functions whose statistics we wish to determine. Due to continuity across the boundaries, the solution is almost surely uniquely determined everywhere for a specified incoming wave. By employing the continuity conditions for the wave field and Eqns. (3) we can restate the boundary value problem for the random medium as follows:

$$L[w]=0, \qquad 0<x<\lambda, \tag{4}$$

with the boundary conditions

$$M[w]|_{x=0}=0, \tag{5a}$$

$$M^*[w]|_{x=\lambda}=-2ik_0, \tag{5b}$$

where M is defined by the operator

$$M=\left(\frac{\partial}{\partial x}+ik_0\right), \tag{6}$$

and M^* is the conjugate complex operator.

Following [3], we can transform the boundary value problem (4)–(5) into a Cauchy type initial value problem by seeking to examine variations of the boundary value Green's function with the domain. The results obtained lead to a set of first order non-linear, coupled stochastic differential equations of the Riccati type:

$$\frac{\partial G_{11}}{\partial\lambda}=1+2ik_0G_{11}-(k_0^2-k^2(\lambda))G_{11}^2, \tag{7a}$$

$$\frac{\partial G_{12}}{\partial\lambda}=ik_0G_{12}-(k_0^2-k^2(\lambda))G_{11}G_{12}, \tag{7b}$$

$$\frac{\partial G_{22}}{\partial\lambda}=-(k_0^2-k^2(\lambda))G_{12}^2, \tag{7c}$$

with the initial conditions

$$G_{kl}(0)=i/2k_0, \qquad k,l=1,2, \tag{8}$$

where $G(x,\xi;\mu)$ defines the random Green's function associated with the boundary value problem (4)–(5); the G_{kl}'s, k, $l=1$, 2, define the boundary values of the domain Green's function, i.e.,

$$\begin{aligned} G_{11}(\lambda)&=G(\lambda,\lambda),\\ G_{12}(\lambda)&=G(0,\lambda)=G(\lambda,0)=G_{21}(\lambda),\\ G_{22}(\lambda)&=G(0,0), \end{aligned} \tag{9}$$

and the wave field is related to $G(x,\xi;\mu)$ by the relationship

$$w(x,\lambda;\mu)=-2ik_0G(\lambda,x;\mu). \tag{10}$$

In order to determine the complete statistical characteristics of R and T, we need determine the probabilistic solutions for G_{11} and G_{12}. As we are mainly interested in the mean power reflected and transmitted, by the conservation of energy, a knowledge of the mean-power reflected uniquely determines the mean power transmitted and consequently, we need only calculate the probabilistic nature of G_{11}. We shall, therefore, concentrate only on the solution associated with Eqns. (7a) and (8).

By the continuity of the wave field across $x=\lambda$, and by the use of Eqns. (3a) and (10) it follows that

$$-2\mathrm{i}k_0 G = 1+R\,, \tag{11}$$

where we have dropped the subscript of G_{11}. Substituting Eqn. (11) into Eqns. (7a) and (8) implies the equation for the reflected amplitude, namely,

$$\frac{\partial R}{\partial \lambda} = 2\mathrm{i}k_0 R + \frac{\mathrm{i}\beta k_0 \gamma(\lambda;\mu)}{2}(1+R)^2 \tag{12}$$

with the initial condition

$$R(0) = 0\,. \tag{13}$$

It is convenient to let

$$R(\lambda;\mu) = r(\lambda;\mu)\mathrm{e}^{\mathrm{i}\theta(\lambda;\mu)}\,, \qquad 0\leqslant r\leqslant 1\,,\ 0\leqslant\theta\leqslant 2\pi\,, \tag{14}$$

so as to obtain equations in the real random variables r and θ. Upon substituting Eqn. (14) into Eqn. (12), adding and subtracting conjugate complex equations, we arrive at the following two stochastic equations for the magnitude and the phase of the reflected amplitude, respectively:

$$\frac{\partial r}{\partial\lambda} = \frac{\beta k_0\gamma(\lambda;\mu)}{2}(1-r^2)\sin(\phi+2k_0\lambda)\,, \tag{15a}$$

$$\frac{\partial \phi}{\partial\lambda} = \frac{\beta k_0\gamma(\lambda;\mu)}{2}\left[2+\frac{(1+r^2)}{r}\cos(\phi+2k_0\lambda)\right], \tag{15b}$$

where

$$\phi = \theta - 2k_0\lambda\,. \tag{16}$$

Equations (15a) and (15b) constitute the appropriate differential equations from which the Fokker–Planck equation, or Kolmogorov's forward equation can be constructed for an appropriate process $\gamma(\lambda;\mu)$. We should note that these equations are valid for arbitrary stochastic processes $\gamma(\lambda;\mu)$ and that the fluctuations in the field need not be small.

3. Fokker–Planck's equation and the averaged equation

Let us construct the equation for the conservation of probability when $\gamma(\lambda;\mu)$ is a white-Gaussian process and when the fluctuations about the mean of the random scattering medium are considered small, i.e., choose

$$\begin{aligned} &E\{\gamma(\lambda)\} = 0\,,\\ &E\{\gamma(\lambda_1)\gamma(\lambda_2)\} = \sigma^2\delta(\lambda_1-\lambda_2)\,, \end{aligned} \tag{17}$$

and

$$\beta = \varepsilon \ll 1\,, \tag{18}$$

where $E\{\cdot\}$ defines taking the expected values, i.e., $E\{\cdot\} = \int_\Omega \{\cdot\}\,dP(\mu)$.

The process $\gamma(\lambda;\mu)$ is delta-correlated, and therefore, is not mean-square Riemann integrable. Consequently, we need consider Eqns. (15a) and (15b) in the Itô sense [13]. Since the white-Gaussian process is the formal derivative of the Wiener process (or Brownian motion), i.e. $\gamma(\lambda)\sim dB(\lambda)/d\lambda$ we may write Eqns. (15) in the equivalent form of

$$dr = \frac{\varepsilon k_0}{2}(1-r^2)\sin(\phi+2k_0\lambda)\,dB(\lambda)\,, \tag{19a}$$

$$d\phi = \frac{\varepsilon k_0}{2}\left[2+\frac{(1+r^2)}{r}\cos(\phi+2k_0\lambda)\right]dB(\lambda)\,, \tag{19b}$$

where the Wiener process has mean zero and

$$E\{B(\lambda_1)B(\lambda_2)\} = \sigma^2 \min(\lambda_1, \lambda_2), \quad \lambda_1, \lambda_2 > 0, \tag{20}$$

and its sample functions are continuous with probability one and are of unbounded variation with probability one.

Suppose now we consider an auxiliary system of stochastic differential equations such that their limiting solutions when a certain parameter approaches zero tend to the solutions of Eqns. (19a) and (19b). We consider this auxiliary system because Eqns. (19a) and (19b) lead to a diffusion operator some of whose coefficients are not uniformly continuous for $\lambda \to 0$, in which case Khas'minskii's averaging theorem is not applicable. In particular, let

$$dr_{(\delta)} = \frac{\varepsilon k_0}{2}(1-r_{(\delta)}^2)\sin(\phi_{(\delta)}+2k_0\lambda)\,dB(\lambda), \tag{21a}$$

$$d\phi_{(\delta)} = \frac{\varepsilon k_0}{2}\left[2 + \frac{(1+r_{(\delta)}^2)}{(r_{(\delta)}+\delta)}\cos(\phi_{(\delta)}+2k_0\lambda)\right] dB(\lambda), \tag{21b}$$

where $\delta \geqslant 0$ varies through the bounded parameter set Δ. In addition, we will assume for the initial conditions that $r_{(\delta)}(0) = r(0)$ and $\phi_{(\delta)}(0) = \phi(0)$. By studying the dependence of the solutions $r_{(\delta)}(\lambda)$ and $\phi_{(\delta)}(\lambda)$ on the parameter δ, it can be shown [14] that $r_{(\delta)}$ and $\phi_{(\delta)}$ are continuous in mean-square with respect to δ (certain conditions need be satisfied), and moreover,

$$\lim_{\delta\to 0}\sup_{\lambda} E\{[r_{(\delta)}(\lambda)-r(\lambda)]^2\} = 0,$$
$$\lim_{\delta\to 0}\sup_{\lambda} E\{[\phi_{(\delta)}(\lambda)-\phi(\lambda)]^2\} = 0.$$

Similar results hold for the differentiability of $r_{(\delta)}$ and $\phi_{(\delta)}$ with respect to the parameter δ. This continuous dependence of the solutions on δ shall be exploited to obtain the appropriate averaged diffusion operator associated with Eqns. (19a) and (19b).

The Fokker–Planck equation associated with the auxiliary system can now be written down immediately as:

$$\frac{\partial P^{(\delta)}(r, \phi, \lambda)}{\partial \lambda} = \mathscr{L}_{(\delta)}(P^{(\delta)}), \tag{22}$$

where $\mathscr{L}_{(\delta)}$ is the forward diffusion operator associated with the Markov processes $r_{(\delta)}(\lambda;\mu)$ and $\phi_{(\delta)}(\lambda;\mu)$ which are generated by the Itô type equations (21a) and (21b):

$$\mathscr{L}_{(\delta)}(\cdot) = \frac{\varepsilon^2 k_0^2 \sigma^2}{8}\left(\frac{\partial^2}{\partial r^2}[A(r,\phi,\lambda)\cdot] + \frac{\partial^2}{\partial r\,\partial\phi}[B(r,\phi,\lambda)\cdot] + \frac{\partial^2}{\partial\phi^2}[C(r,\phi,\lambda)\cdot]\right). \tag{23}$$

where

$$A(r,\phi,\lambda) = (1-r^2)^2\sin^2(\phi+2k_0\lambda),$$
$$B(r,\phi,\lambda) = 2(1-r^2)\sin(\phi+2k_0\lambda)\left[2 + \frac{(1+r^2)}{(r+\delta)}\cos(\phi+2k_0\lambda)\right], \tag{24}$$
$$C(r,\phi,\lambda) = \left[2 + \frac{(1+r^2)}{(r+\delta)}\cos(\phi+2k_0\lambda)\right]^2.$$

In Eqns. (22), (23) and (24), we have suppressed the subscript δ on r and ϕ. For the application of the averaging principle, it is convenient to write Eqn. (22) in the following form:

$$\frac{\partial P^{(\delta)}}{\partial\lambda} = \frac{\varepsilon^2 k_0^2\sigma^2}{8}(AP_{rr}^{(\delta)} + BP_{r\phi}^{(\delta)} + CP_{\phi\phi}^{(\delta)} + DP_r^{(\delta)} + EP_\phi^{(\delta)} + FP^{(\delta)}), \tag{25}$$

where

$$D = 2A_r + B_\phi,$$
$$E = B_r + 2C_\phi, \tag{26}$$
$$F = A_{rr} + B_{\phi r} + C_{\phi\phi}.$$

Let us now consider the principle of averaging for the solution of the Cauchy problem [1] in a convenient form for our use. We remark first that the theorem itself is a study of the limit behavior of the diffusion equation whose coefficients oscillate rapidly. We can see this by setting $s=\varepsilon^2\lambda$ in Eqn. (25), in which case, all the coefficients become functions of $\varepsilon^{-2}s$, for example, $A=A(r, \phi, \varepsilon^{-2}s)$, etc., and then we ask for the limiting solution as $\varepsilon\to 0$.

Let $I_L=(0, L)$. We shall consider the following equation in the region $D\times I_\infty$, where $D=\{(x_1, x_2): x_1\in[0, 1], x_2\in(-\infty, 2\pi]\}$:

$$\frac{\partial P}{\partial\lambda}=\varepsilon^2\left[\sum_{i,j=1}^{2} a_{ij}(\bar{x}, \lambda)\frac{\partial^2 P}{\partial x_i\partial x_j}+\sum_{i=1}^{2} b_i(\bar{x}, \lambda)\frac{\partial P}{\partial x_i}+c(\bar{x}, \lambda)P+d(\bar{x}, \lambda)\right], \tag{27}$$

where $\bar{x}=(x_1, x_2)$. In addition, we shall assume that the following conditions are satisfied by the coefficients of the Eqn. (27):

(i) $\|a_{ij}\|$ is non-negative definite in $(\bar{x}, \lambda)\in D\times I_\infty$. a_{ij}, b_i, c and d are continuous in its arguments, are bounded for $\lambda>0$ and are sufficiently smooth so that a solution to Eqn. (27) exists ($\varepsilon>0$).

(ii) a_{ij}, b_i, c and d are uniformly continuous in $\bar{x}$ in $(\bar{x}, \lambda)\in D\times I_\infty$.

(iii) the limits of the means* with respect to λ as $\lambda\to\infty$ exists for a_{ij}, b_i, c and d, uniformly in $\bar{x}\in D$.

(iv) a solution to the following averaged equation exists:

$$\frac{\partial P}{\partial\lambda}=\sum_{i,j=1}^{2}\langle a_{ij}(\bar{x})\rangle\frac{\partial^2 P}{\partial x_i\partial x_j}+\sum_{i=1}^{2}\langle b_i(\bar{x})\rangle\frac{\partial P}{\partial x_i}+\langle c(\bar{x})\rangle P+\langle d(\bar{x})\rangle\,. \tag{28}$$

Khas'minskii's theorem states:

Let conditions (i)–(iv) *be satisfied and let* $P_\varepsilon(\bar{x}, \lambda)$ *be the solution of Eqn.* (27) *in the region* $D\times I_{L/\varepsilon^2}$, *satisfying the condition*

$$\lim_{\lambda\to L/\varepsilon^2} P_\varepsilon(\bar{x}, \lambda)=F(\bar{x})\,, \tag{29}$$

where $F(\bar{x})$ *is a continuous bounded function in* D. *Furthermore, let* $\tilde{P}(\bar{x}, \lambda)$ *be a solution of Eqn.* (28) *in the region* $D\times I_L$, *satisfying* $\tilde{P}(\bar{x}, L)=F(\bar{x})$. *Then*

$$\lim_{\varepsilon\to 0}\sup_{(\bar{x}, \lambda)\in D\times I_L}\left|P_\varepsilon\left(\bar{x}, \frac{\lambda}{\varepsilon^2}\right)-\tilde{P}(\bar{x}, \lambda)\right|=0\,. \tag{30}$$

We can now apply the above theorem to the solution of Eqn. (25). Let $\delta>0$. The only condition not satisfied due to a degeneracy of the matrix $\|a_{ij}\|$ is the first part of (i). We overcome this problem by introducing into a_{jj}, $j=1$ or 2, a sufficiently smooth and rapidly oscillating function $\psi(\lambda)$, such that it satisfies the conditions: (i) $\langle\psi\rangle=0$, and (ii) $a_{jj}(\bar{x}, \lambda)+\alpha\psi(\lambda)\geqslant a^*>0$, $\alpha>0$ a parameter, and hence $\|a_{ij}\|$ positive definite. In such a way all the conditions are met. The proof for this is not complicated and will not be dealt with here, as we are more interested in obtaining the statistical properties of the wave. We now average Eqn. (25) with $P_\alpha^{(\delta)}$ (associated with $a_{jj}+\alpha\psi$, $j=1$ or 2) and exploit the continuous dependence of $\tilde{P}_\alpha^{(\delta)}$ on α to obtain $\tilde{P}_\alpha^{(\delta)}\to\tilde{P}^{(\delta)}$.

Performing the analysis leads to the equation

$$\frac{\partial\tilde{P}(r, \tau)}{\partial\tau}=\langle\mathscr{L}_m(\tilde{P})\rangle\,, \tag{31}$$

where $\langle\mathscr{L}_m(\cdot)\rangle$ denotes the averaged marginal generator in Eqn. (28) involving r only, and is given by

$$\langle\mathscr{L}_m(\cdot)\rangle=\frac{\partial^2}{\partial r^2}[(1-r^2)^2(\cdot)] \tag{32}$$

* The limit of the mean is defined by

$$\langle g(\bar{x})\rangle=\lim_{L\to\infty}\frac{1}{L}\int_0^L g(\bar{x}, \lambda)d\lambda\,.$$

and

$$\tau = \frac{\varepsilon^2 k_0^2 \sigma^2}{16} \lambda . \tag{33}$$

In deriving Eqn. (32), we have integrated out from the generator in Eqn. (28) the ϕ dependence, ϕ being the phase of the reflection amplitude; moreover, we have used the continuous dependence of $r_{(\delta)}$ and $\phi_{(\delta)}$ on the parameter δ, and hence the continuous dependence of $P^{(\delta)}$ on δ, to take the limit $\tilde{P}^{(\delta)} \to \tilde{P}$ as $\delta \to 0$. Since the phase ϕ is not determined by the solution of Eqn. (31), we can conclude that the values of the phase, $-2k_0\lambda < \phi < 2\pi - 2k_0\lambda$, are equi-probable. The solution to Eqn. (31) will be uniformly valid on the interval $\lambda \to L/\varepsilon^2$, $\varepsilon \to 0$.

4. Mean power reflected and transmitted

First, let us determine what initial condition is associated with Eqn. (31). From physical considerations, for $r=0$, no energy is reflected and so the appropriate initial condition is given by

$$\tilde{P}(r, 0) = \delta(r) . \tag{34}$$

By obtaining the solution $\tilde{P}(r, \tau)$ associated with Eqns. (31)–(34), i.e., the approximate transition probability density function for small ε, we can determine the complete statistical characteristics or r, where r is defined in Eqn. (14). In particular, we shall be interested in the mean power reflected defined by

$$E\{|R(\lambda; \mu)|^2\} = E\{r^2\} = \int_0^1 r^2 \tilde{P}(r, \tau) \mathrm{d}r . \tag{35}$$

The mean power transmitted can be obtained from the conservation of energy equation and by the use of Eqn. (35), namely,

$$E\{|R(\lambda; \mu)|^2\} + E\{|T(\lambda; \mu)|^2\} = 1 .$$

Let us consider the following transformation of variables:

$$r = f(u) = \tanh u , \quad \tilde{P}(r, \tau) = g(u, \tau) . \tag{36}$$

It follows that

$$u = f^{-1}(r) = \tfrac{1}{2} \ln \left(\frac{1+r}{1-r} \right), \quad u \geqslant 0 . \tag{37}$$

Upon substitution of the above transformation into Eqns. (31) and (34) we arrive at the equation

$$\frac{\partial \tilde{G}}{\partial \tau} = \frac{1}{\eta(u)} \frac{\partial}{\partial u} \left[\eta(u) \frac{\partial \tilde{G}}{\partial u} \right], \tag{38}$$

where we have set

$$\tilde{G}(u, \tau) = \operatorname{sech}^4 u \, g(u, \tau) \tag{39}$$

and

$$\eta(u) = \cosh^2 u . \tag{40}$$

It is apparent that the initial condition on $\tilde{G}(u, \tau)$ should be

$$\tilde{G}(u, 0) = \delta(u) . \tag{41}$$

The right-hand side of Eqn. (38) can be recognized as the Beltrami–Laplace operator. By analogy to the formulation [15], we obtain the solution to Eqns. (38) and (41). Upon utilizing this solution and Eqn. (39) we can write the expression for $g(u, \tau)$ as

$$g(u, \tau) = \frac{1}{2(\pi^{\frac{1}{2}})} \mathrm{e}^{-\tau} \tau^{-\frac{3}{2}} u \cosh^3 u \, \mathrm{e}^{-u^2/4\tau} \geqslant 0 . \tag{42}$$

By the use of Eqns. (35), (36) and (42) we can now write an expression for the approximate mean power reflected, namely,

$$E\{r^2\} = \int_0^\infty \tanh^2 u\, g(u, \tau)\, df(u)$$

$$= \frac{1}{2(\pi^{\frac{1}{2}})} e^{-\tau} \tau^{-\frac{3}{2}} \int_0^\infty u \sinh u \tanh u\, e^{-u^2/4\tau} du\,. \tag{43}$$

By the conservation relation we also obtain an approximate expression for the mean power transmitted

$$E\{t^2\} = E\{|T(\lambda;\, u)|^2\}$$

$$= 1 - \frac{1}{2(\pi^{\frac{1}{2}})} e^{-\tau} \tau^{-\frac{3}{2}} \int_0^\infty u \sinh u \tanh u\, e^{-u^2/4\tau} du\,. \tag{44}$$

Let us now note the following two things: (i) $\tilde{P}$ is an approximate density function, and as such, it should satisfy $\int_0^1 \tilde{P}\, dr \approx 1$ for an appropriate domain of τ. Utilizing Eqns. (36) and (42) in the normalization condition and performing the integration implies

$$\int_0^\infty g(u, \tau) f'(u) du = \frac{e^{-\tau}}{(\pi\tau)^{\frac{1}{2}}} + \operatorname{erf} \tau^{\frac{1}{2}}\,,$$

which clearly gives in the limiting case $\tau \to \infty$, $\int_0^\infty g f' du \to 1$; (ii) the approximate density function is in fact the exact density function relative to the measure $m(u) = \ln \cosh u$, so that, $\int_0^\infty g(u, \tau) f'(u) dm(u) = 1$, for all $\tau \geqslant 0$.

In Fig. 1, we present a graph of Eqn. (43) which was obtained by numerical integration. In addition, we compare this result with that of Papanicolaou [8] where we have set $s = \frac{1}{2} k_0^2 \sigma^2$ in Eqn. (3.8), and with the quasilinearization result [3], where

$$E\{r^2\} = 1 - e^{-4\tau} + O(\tfrac{1}{16}\varepsilon^2 k_0^2 \sigma^2 e^{-2\tau}) \text{ and we plot } E\{r^2\}_{min} = 1 - e^{-4\tau}\,.$$

If we consider in Eqns. (15) $\gamma(\lambda;\, \mu)$ to be a centered bounded real stochastic process with covariance function $E\{\gamma(\lambda_1)\gamma(\lambda_2)\} = \Gamma(|\lambda_1 - \lambda_2|)$ and $\gamma(\lambda;\, \mu)$ satisfying a strong mixing condition, then the solution $P(r, \lambda)$ is given by Eqn. (3.10) of [8], where now $r = (u-1)^{\frac{1}{2}}(u+1)^{-\frac{1}{2}}$.

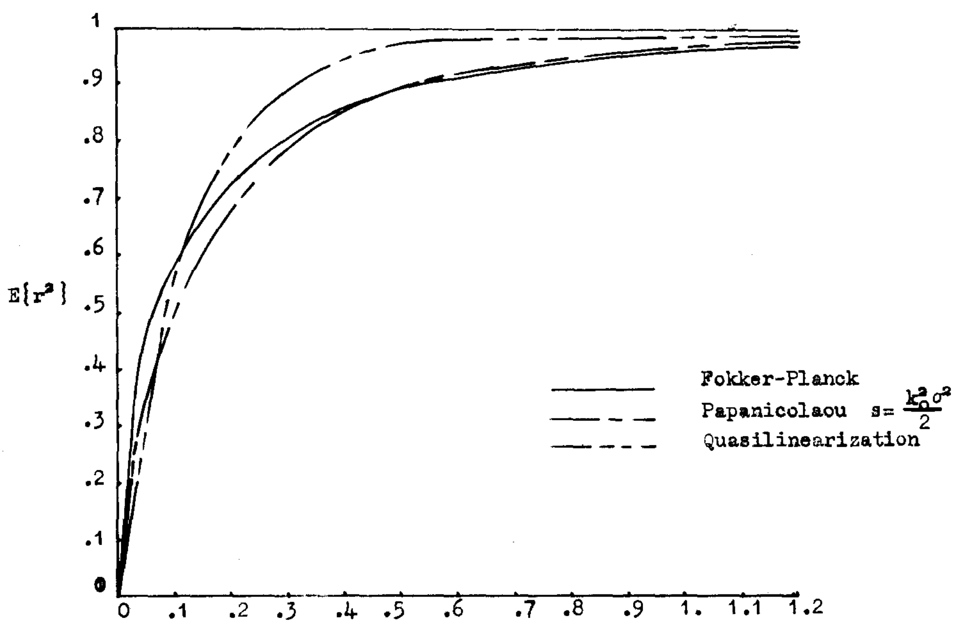

Figure 1. Approximate mean power reflected as a function of τ where $\tau = \frac{1}{16}(\varepsilon^2 k_0^2 \sigma^2)\lambda$.

REFERENCES

[1] R. Z. Khas'minskii, Principle of Averaging for Parabolic and Elliptic Differential Equations and for Markov Processes with Small Diffusion, *Theory of Probability and its Applications*, III, 1 (1963) 1–21.
[2] E. Ghandour, Initial Value Problem for Boundary Values of a Green's Function, *SIAM J. Appl. Math.*, 27, 4 (1974).
[3] E. Ghandour, One Dimensional Random Wave Propagation, To appear in *SIAM J. Appl. Math.*
[4] J. L. Sanders, Jr., Unpublished notes.
[5] J. L. Sanders, Jr., Random Deflections of a String on an Elastic Foundation, *SIAM J. Appl. Math.*, 22, 3 (1972) 406–418.
[6] G. F. Carrier, *Stochastically Driven Dynamical Systems*, Harvard University (1969).
[7] J. A. Morrison, G. C. Papanicolaou and J. B. Keller, Mean Power Transmission through a Slab of Random Medium, *Comm. on Pure and Appl. Math.*, XXIV (1971) 473–489.
[8] G. C. Papanicolaou, Wave Propagation in a One-Dimensional Random Medium, *SIAM J. Appl. Math.*, 21, 1 (1971) 13–18.
[9] U. Frish, Wave Propagation in a Random Media, in *Probabilistic Methods in Applied Mathematics*, Vol. 1, Bharucha-Reid (editor), Academic Press, New York (1968).
[10] I. Kay and R. A. Silverman, Multiple Scattering by a Random Stack of Dielectric Slabs, *Nuovo Cimento*, 9 Suppl. (1958) 626–645.
[11] R. Z. Khas'minskii, On Stochastic Processes Defined by Differential Equations with a Small Parameter, *Theory of Probability and its Applications*, XI, 2 (1966) 211–228.
[12] R. Z. Khas'minskii, A Limit Theorem for the Solutions of Differential Equations with Random Right-Hand Sides, *Theory of Probability and its Applications*, XI, 3 (1966) 390–406.
[13] I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, Saunders (1969).
[14] I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations*, Springer-Verlag (1972) 50–59.
[15] F. I. Karpelevich, V. N. Tutubalin and M. G. Shur, Limit Theorems for the Compositions of Distributions in the Lobachevsky Plane and Space, *Theory of Probability and its Applications*, IV, 4 (1959).